Pressure-robustness – a new criterion for the accuracy of incompressible Navier-Stokes solvers at high Reynolds numbers and beyond

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- Nicolas Gauger, TU Kaiserslautern

Software: NGSOLVE (J. Schöberl, C. Lehrenfeld)

BMBF-VIP+ Proposal

Proposal in funding line VIP+ (‚Validation of technological potential of innovative science‘) from Federal Ministry of Education and Research (BMBF)

Tentative partners for implementation in SU2:

- Nicolas Gauger, TU Kaiserslautern
- Alexander Linke, Weierstrass Institute, Berlin
- Cornelia Grabe, German Aerospace Center (DLR)

Collaborators (to be confirmed):

- Edwin van der Weide, U Twente
- Martin Schifko, Engineering Software Steyr (ESS)

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Main references

improved understanding of steady Stokes & beyond

V. John, A. L., C. Merdon, M. Neilan, L. Rebholz: On the divergence constraint in mixed FEM for incompressible flows. SIAM Review, Vol. 59(3), 2017.

N. Gauger, P. Schroeder, A. Linke: On high-order pressure-robust space discretisations, their advantages for incompressible high Reynolds number generalised Beltrami flows and beyond. arXiv 1808.10711.

improved understanding of (laminar) transient high Reynolds number Navier-Stokes

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Outline

- 3 examples: pressure-robust vs. non-pressure-robust solvers
- original sin of incompressible CFD: a relaxed $\mathbf{L}^2$-orthogonality
- material derivative in incompressible Euler flows

pressure-robustness inside:

new seal of quality for incompressible/low Mach number CFD

Incompressible Navier-Stokes equations (iNSE)

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}$$
$$\nabla \cdot \mathbf{u} = 0$$

- iNSE in primitive variables
- space discretization at high Reynolds numbers, $0 < \nu \ll 1$

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Example 1: Moving Gresho vortex

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0}$$

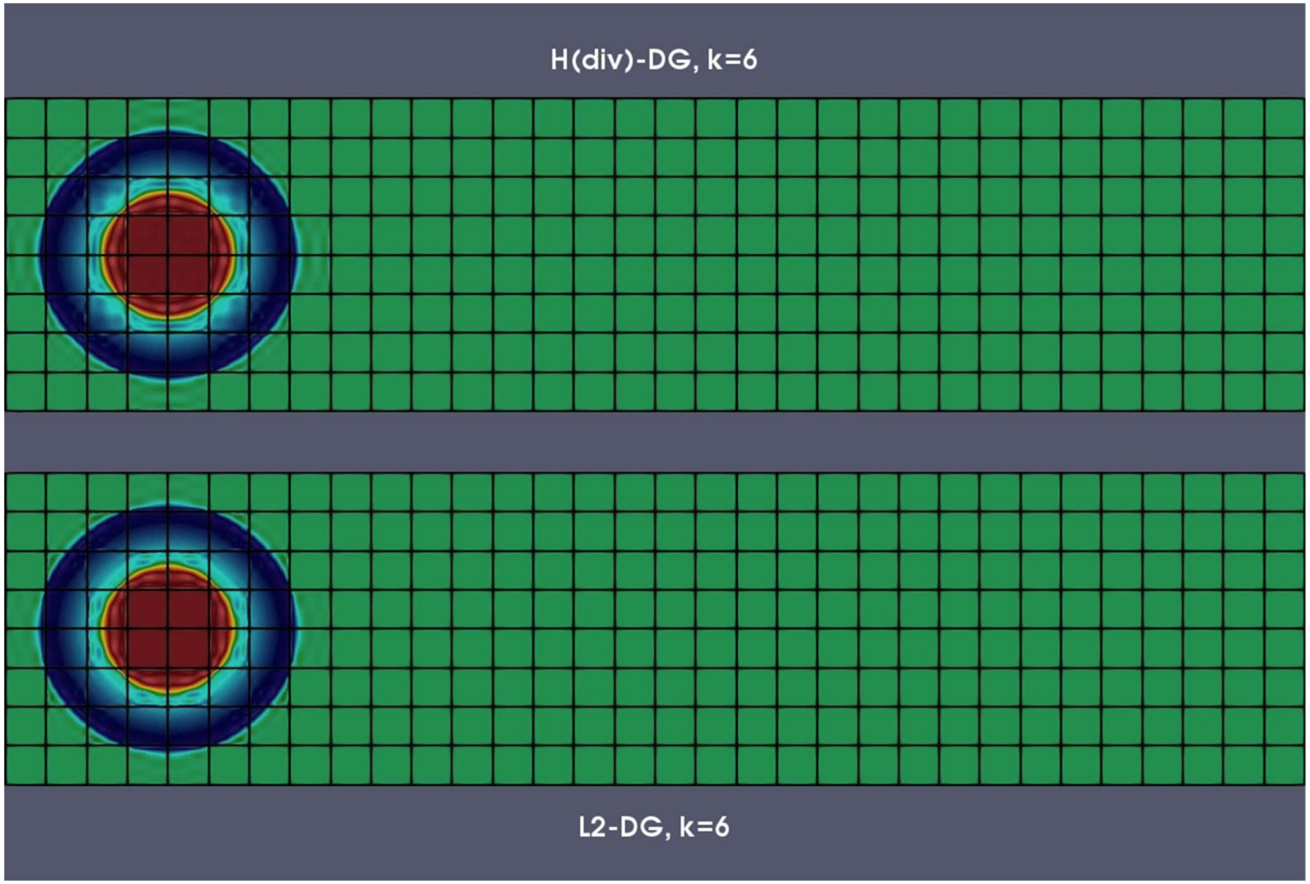
$$\nabla \cdot \mathbf{u} = 0$$

$$\boxed{\nu = 10^{-5}, \qquad t \in (0, 15]}$$

- nontrivial Reynolds number
- dominant nonlinear convection & nontrivial initial value

Example 1: Moving Gresho vortex

$$\mathbb{RT}_6 - \mathbb{P}_5^{\mathrm{dc}} + \text{ upwind} \qquad \text{vs.} \qquad \mathbb{P}_6^{\mathrm{dc}} - \mathbb{P}_5^{\mathrm{dc}} + \text{ upwind}$$



Example 1: Moving Gresho vortex

$$\underbrace{\mathbb{RT}_6 - \mathbb{P}_5^{\mathrm{dc}}}_{\text{pressure-robust}} \quad \text{vs.} \quad \mathbb{P}_6^{\mathrm{dc}} - \mathbb{P}_5^{\mathrm{dc}}$$

References (Philipp W. Schroeder):

- PhD thesis, U Göttingen, 2019.
- www.youtube.com/watch?v=wrZTUrGxVSc

Why pressure-robust DG method more accurate?

A warning

- talk not about mass conservation: velocity trial functions
- but pressure-robustness: velocity test functions
- confusion in Galerkin setting: trial functions = test functions

Reference: A. Linke, C. Merdon: Pressure-robustness [...]. CMAME 2016.

Example 2: Planar lattice flow

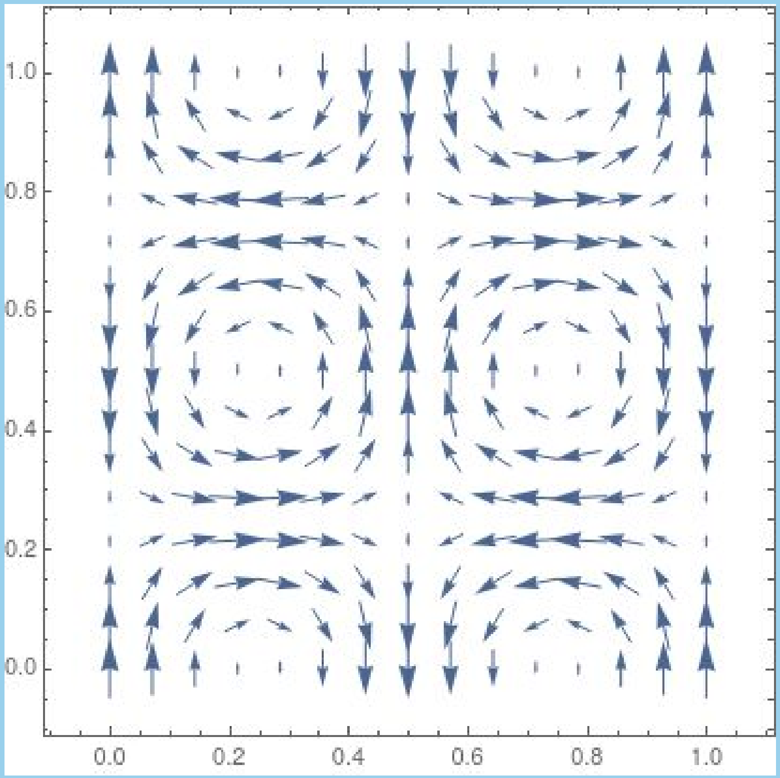
$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\boxed{\nu = 10^{-5}, \qquad t \in (0, 10]}$$

- nontrivial Reynolds number
- dominant nonlinear convection & nontrivial initial value

Example 2: Planar lattice flow



$$\mathbf{u}_0(\mathbf{x}) = \begin{pmatrix} \sin(2\pi x)\sin(2\pi y) \\ \cos(2\pi x)\cos(2\pi y) \end{pmatrix},$$

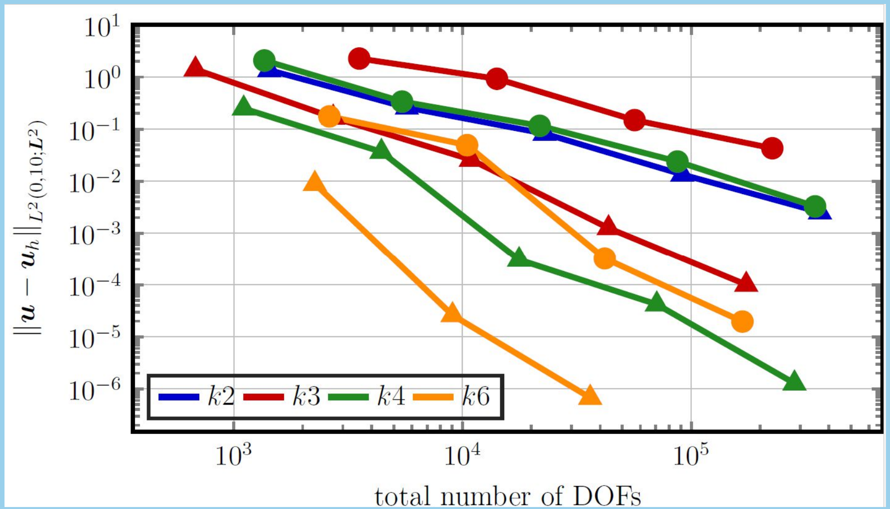
$$\mathbf{u}(t,\mathbf{x}) = \mathbf{u}_0(\mathbf{x})e^{-8\pi^2\nu t}$$

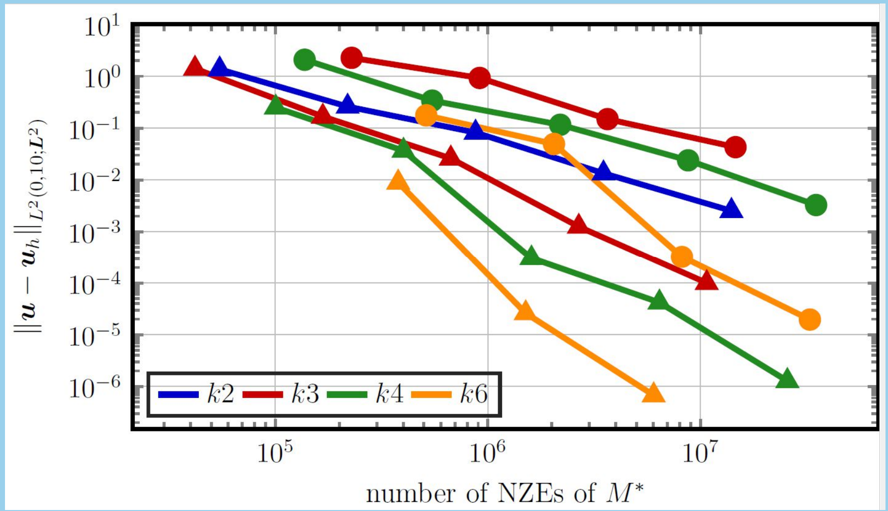
$$p_0(\mathbf{x}) = \frac{1}{4}\left(\cos(4\pi x) - \cos(4\pi y)\right),$$

$$p(t,\mathbf{x}) = p_0(\mathbf{x})e^{-16\pi^2\nu t}$$

- nontrivial Reynolds number
- dominant nonlinear convection & nontrivial initial value

Example 2: Planar lattice flow





$$\underbrace{\mathbb{BDM}_k - \mathbb{P}_{k-1}^{\mathrm{dc}}}_{\text{pressure-robust}} \qquad \text{vs.} \qquad \mathbb{P}_k^{\mathrm{dc}} - \mathbb{P}_{k-1}^{\mathrm{dc}}$$

- pressure-robust solvers (triangles) outperform non-pressure-robust ones (circles)
- coarse grids: non-pressure-robust solvers lose half of (formal) convergence order

Example 3: Steady Stokes flow

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\boxed{\nu = 10^{-3}, \mathbf{u} \in \mathbf{H}_0^1}$$

- nontrivial forcing f
- small viscosity

Example 3: Steady Stokes flow

$$\xi = x^2(1-x)^2 y^2 (1-y)^2$$
$$\mathbf{u} = \mathbf{curl}\,\xi$$
$$p = x^3 + y^3 - \frac{1}{2}$$

$$\boxed{\mathbf{f} = -\nu\Delta\mathbf{u} + \nabla p}$$

- small viscosity
- manufactured f: nearly gradient field

Classical CR-FEM	Pressure-robust CR-RT_0-FEM

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Computer methods in applied mechanics and engineering

On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime

CrossMark

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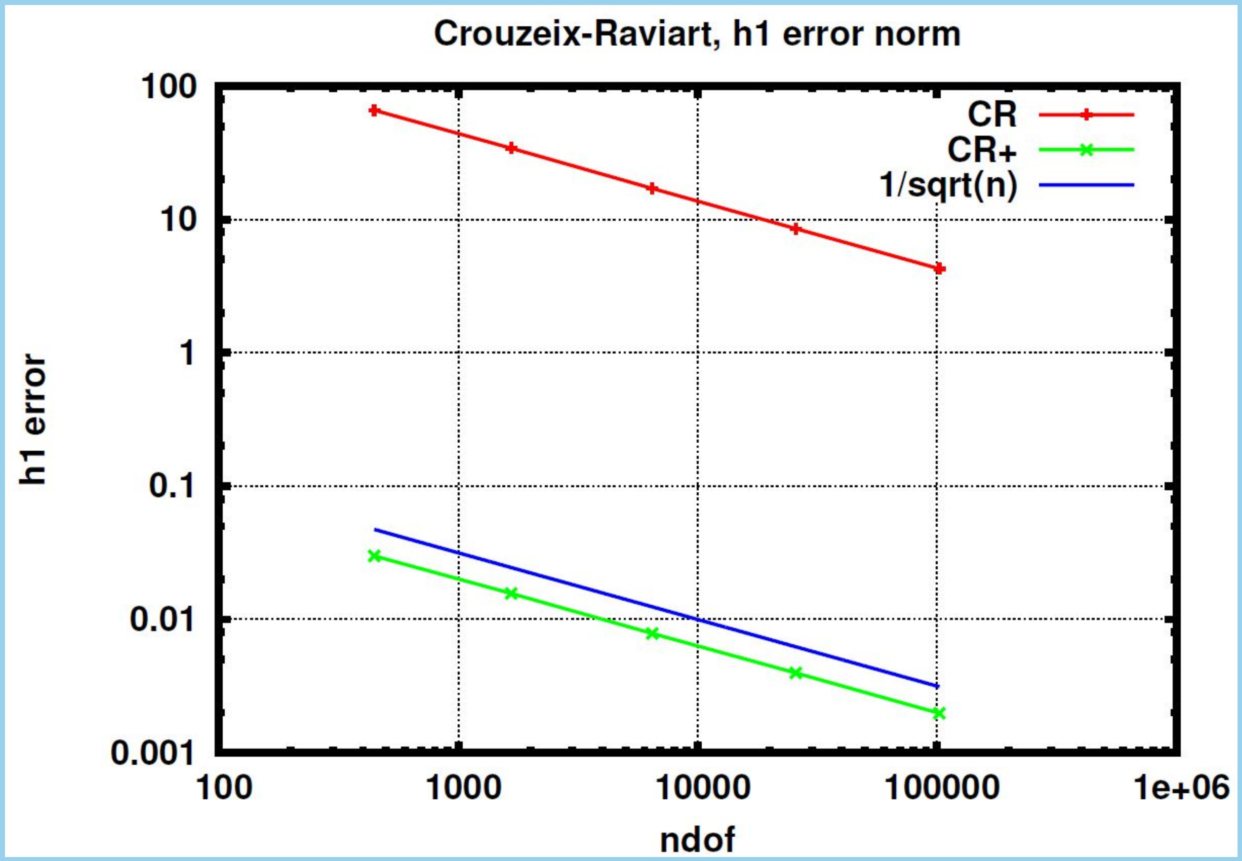
$$\int \mathbf{f} \cdot \mathbf{v}_h \, \mathrm{dx}$$

classical discrete forcing

$$\int \mathbf{f} \cdot I_h^{\mathrm{RT}_0} \mathbf{v}_h \, \mathrm{dx}$$

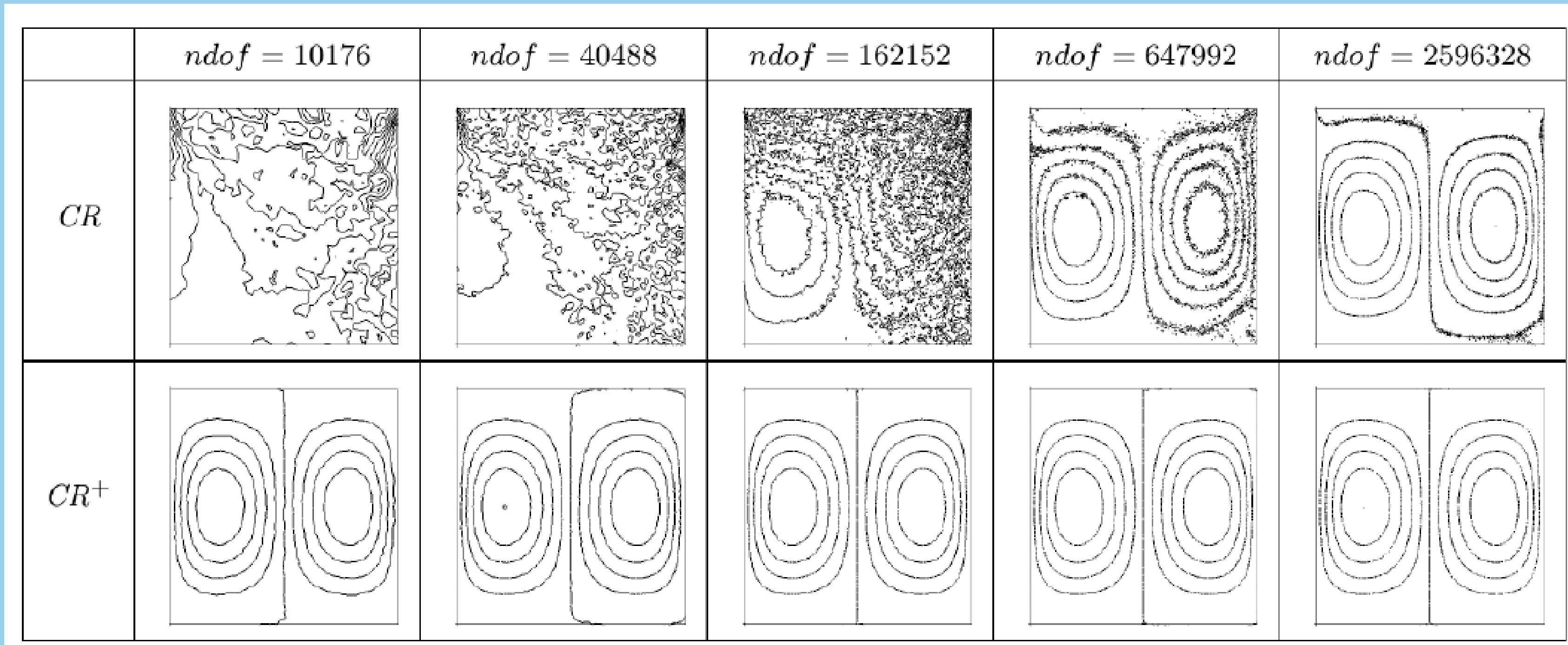
Raviart-Thomas standard interpolator applied elementwise to CR velocity test function

Example 3: Steady Stokes flow



- classical CR-FEM vs. pressure-robust CR-RT_0-FEM
- pressure-robust gain: 10 refinement levels

Example 3: Steady Stokes flow



- new understanding: CR-FEM forcing too strong
- better velocity test functions: performance gains possible

Pressure-robustness vs. non-pressure-robustness

- How to explain dramatic superior accuracy of pressure-robust methods?
- Common reason behind?

Pressure-robustness vs. non-pressure-robustness

Answer:

- pressure-robust methods more robust against dominant gradient fields in momentum balance
- = more robust against strong pressure gradients

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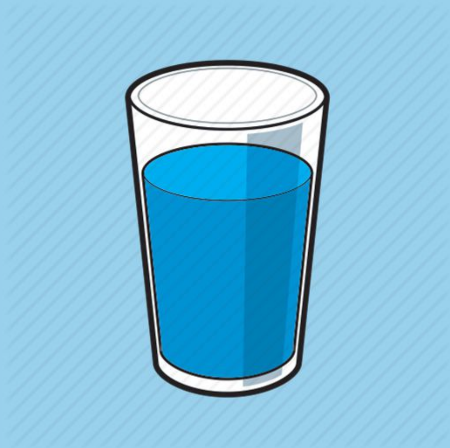
Pressure-robustness vs. non-pressure-robustness

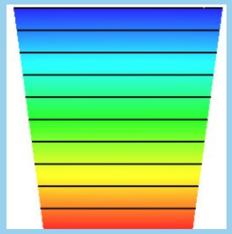
Example	gradient field	momentum balance
Example 1: Gresho vortex	nonlinear convection term	$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{0}$
Example 2: Planar lattice flow	nonlinear convection term	$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{0}$
Example 3: steady Stokes flow	right hand side **f**	$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}$

Reflections on a glass of water - hydrostatics

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nabla\phi$$

$$\nabla\cdot\mathbf{u} = 0$$





Why are gradient fields special? $(\mathbf{u}, \nabla p) = (\mathbf{0}, \nabla\phi)$

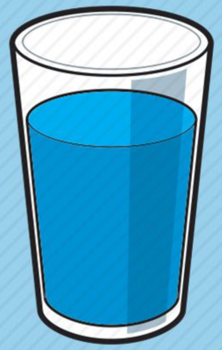
Pressure-robustness vs. non-pressure-robustness

Gradient fields in incompressible Navier-Stokes momentum balance are special:

- they don‘t change velocity
- they only change pressure

Velocity-equivalence of forces

hydrostatics



$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nabla \phi$$

$$\nabla \phi \simeq \mathbf{0}$$

$$\mathbf{f} \simeq \mathbf{g} \quad \Leftrightarrow$$

$$\mathbb{P}(\mathbf{f}) = \mathbb{P}(\mathbf{g}) \quad \Leftrightarrow$$

$$\exists_{\nabla \phi} : \mathbf{f} = \mathbf{g} + \nabla \phi$$

velocity-equivalence induced by semi-norm $\|\mathbb{P}(\mathbf{f})\|_{\mathbf{L}^2}$

Original sin of incompressible/low Mach number CFD

Relaxation of divergence constraint in
- discretely inf-sup stable mixed Stokes methods
- pressure-stabilized mixed Stokes methods

hidden consistency error

relaxed $\mathbf{L}^2$-orthogonality of arbitrary gradient vs. discretely divergence-free velocity test functions

Helmholtz-Hodge decomposition

$$\mathbf{L}^2_\sigma := \{\mathbf{v} \in \mathbf{L}^2 : \int \mathbf{v} \cdot \nabla \phi \, \mathrm{dx} = 0, \quad \text{for all } \phi \in H^1\}!$$

$$\boxed{\mathbf{L}^2 = \mathbf{L}^2_\sigma \bigoplus\nolimits_{\mathbf{L}^2} \nabla(H^1)}$$

- $\mathbf{L}^2_\sigma$: $\mathbf{L}^2$-orthogonal complement to $\mathbf{L}^2$ gradient fields
- major importance in pure mathematics
- key for understanding pressure-robustness

Helmholtz-Hodge projector

$$\mathbf{L}^2_\sigma := \{\mathbf{v} \in \mathbf{L}^2 : \qquad \underbrace{-\int \mathbf{v} \cdot \nabla\phi \, \mathrm{dx} = 0}_{\text{distributional divergence for } \phi \in C_0^\infty}, \quad \text{for all } \phi \in H^1\}!$$

Properties:

- $\mathbf{L}^2_\sigma \subset \mathbf{H}(\mathrm{div})$ weakly divergence-free vector fields
- boundary: zero normal component at boundary

Helmholtz-Hodge projector

$$\mathbf{L}^2_\sigma := \{\mathbf{v} \in \mathbf{L}^2 : \quad \underbrace{-\int \mathbf{v} \cdot \nabla\phi \, \mathrm{dx} = 0}_{\text{distributional divergence for } \phi \in C_0^\infty}, \quad \text{for all } \phi \in H^1\}!$$

Key for pressure-robustness:

- continuous normal component over element faces leads to well-defined divergence
- divergence-free BDM & RT vector fields (boundary: zero normal velocity)

$\mathbf{L}^2$-orthogonality to arbitrary gradient fields !!!

Thanks to F. Brezzi, D. Marini, J. Douglas, P.-A. Raviart, J.-M. Thomas, …

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Helmholtz-Hodge projector

Helmholtz-Hodge projector $\mathbb{P}(\mathbf{f})$: $\mathbf{L}^2$ projector onto $\mathbf{L}^2_\sigma$

Helmholtz-Hodge projector

Fundamental property ($\mathbf{L}^2$-orthogonality):

$$\mathbb{P}(\nabla \phi) = \mathbf{0}$$

Helmholtz-Hodge projector: related to curl operator

Mixed methods

$$\mathbf{L}^2_\sigma \to \mathbf{L}^2_{\sigma,h} \qquad\qquad \mathbb{P} \to \mathbb{P}_h$$

Implicitly defined discretely divergence-free vector fields

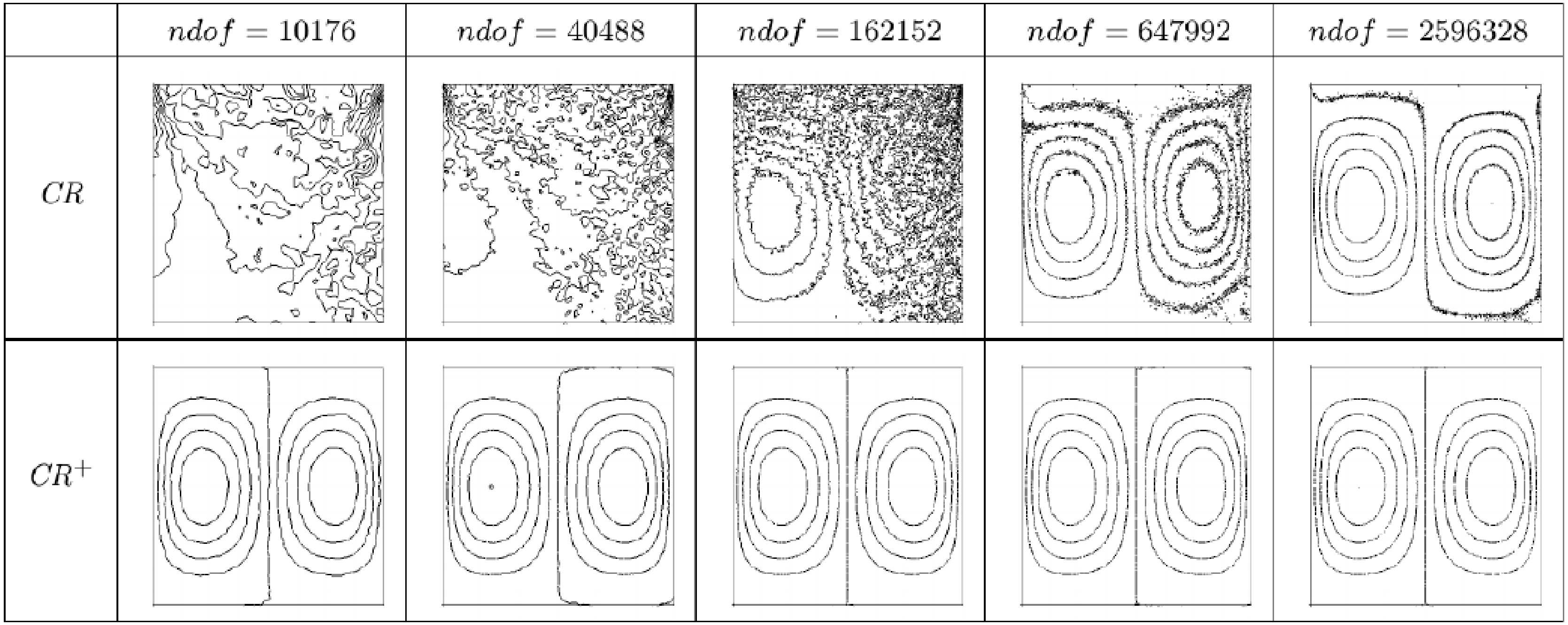
$$\boxed{\mathbf{L}^2_{\sigma,h} := \{\mathbf{v}_h \in \mathbf{V}_h : (\nabla \cdot \mathbf{v}_h, q_h) = 0 \quad \text{for all } q_h \in Q_h\}}$$

Discrete Helmholtz-Hodge projector:

$\mathbf{L}^2$-projection $\mathbb{P}_h$ onto $\mathbf{L}^2_{\sigma,h}$

Non-pressure-robust	Pressure-robust
Example (Taylor-Hood):	Example (Brezzi-Douglas-Marini):
$\mathbb{P}_k - \mathbb{P}_{k-1}$	$\mathbb{BDM}_k - \mathbb{P}^{\mathrm{dc}}_{k-1}$
$\nabla \cdot (\mathbb{P}_k) \supset \mathbb{P}_{k-1}$	$\nabla \cdot (\mathbb{BDM}_k) = \mathbb{P}^{\mathrm{dc}}_{k-1}$
$\mathbf{L}^2_{\sigma,h} \not\subset \mathbf{L}^2_\sigma$	$\mathbf{L}^2_{\sigma,h} \subset \mathbf{L}^2_\sigma$
$\underbrace{\mathbb{P}_h(\nabla p) \neq \mathbf{0}}_{\text{hidden consistency error}}$	$\mathbb{P}_h(\nabla p) = \mathbf{0}$

Example 3: Steady Stokes flow



- new understanding: CR-FEM forcing too strong
- better velocity test functions: performance gains possible

Non-pressure-robust	Pressure-robust

$$-\nu \Delta \mathbf{u} + \nabla p = \mathbf{f}$$
$$\nabla \cdot \mathbf{u} = 0$$

inconsistent data dependence = too strong forcing = large errors

$$\mathbf{u}_h = \left(\mathbb{P}_h \circ (-\Delta_h^{-1}) \circ \mathbb{P}_h\right) \left(\frac{1}{\nu}\mathbf{f}\right)$$
$$= \left(\mathbb{P}_h \circ (-\Delta_h^{-1}) \circ \mathbb{P}_h\right) (-\Delta \mathbf{u})$$
$$+ \left(\mathbb{P}_h \circ (-\Delta_h^{-1}) \circ \mathbb{P}_h\right) \left(\frac{1}{\nu}\nabla p\right)$$

$$\mathbf{u}_h = \left(\mathbb{P}_h \circ (-\Delta_h^{-1}) \circ \mathbb{P}_h\right) \left(\frac{1}{\nu}\mathbb{P}(\mathbf{f})\right)$$
$$= \left(\mathbb{P}_h \circ (-\Delta_h^{-1}) \circ \mathbb{P}_h\right) (\mathbb{P}(-\Delta \mathbf{u}))$$

steady Stokes: T-dependence of velocity error replaced by $\frac{1}{\nu}$ -dependence

Last question – the decisive one

How do dominant pressure gradients develop?

Reference:

N. Gauger, P. Schroeder, A. Linke: arXiv 1808.10711.

Model problem

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{0}$$

$$\nabla \cdot \mathbf{u} = 0$$

model setting

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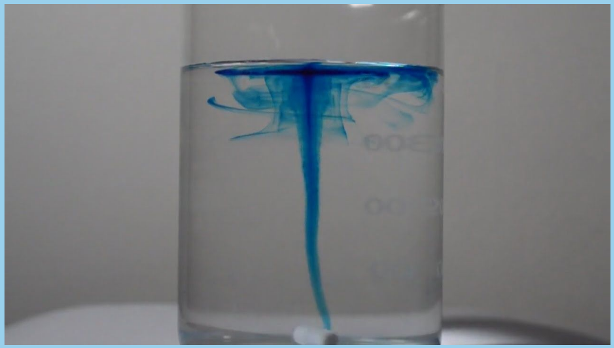
Model problem

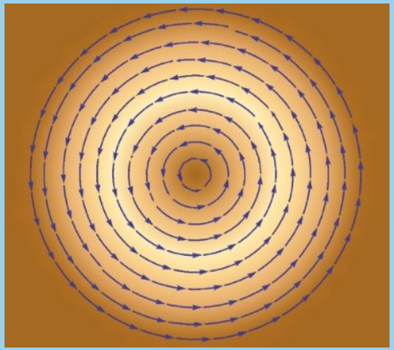
$$\Rightarrow \qquad \frac{D\mathbf{u}}{\mathrm{D}t} := \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p$$

incompressible Euler flow: material derivative – a gradient field!

Model problem – vortex dominated flows

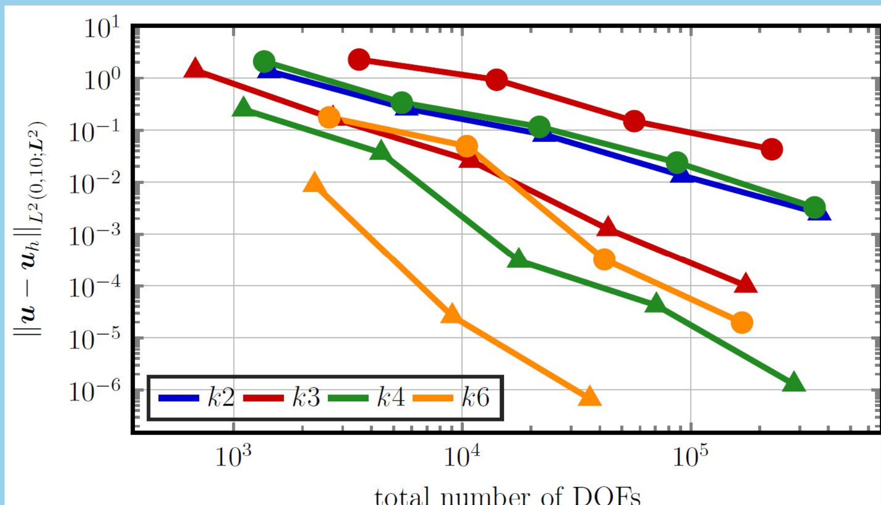
$$\frac{D\mathbf{u}}{\mathrm{Dt}} := \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p$$

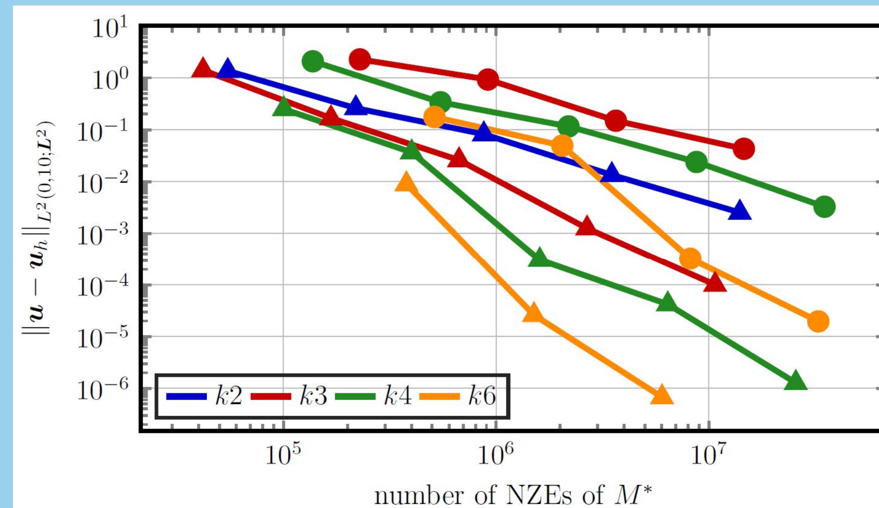




- force balance: centrifugal force = pressure gradient
- quadratic nonlinear convection balances linear pressure gradient
- strong complicated pressure gradient

Example 2: Planar lattice flow



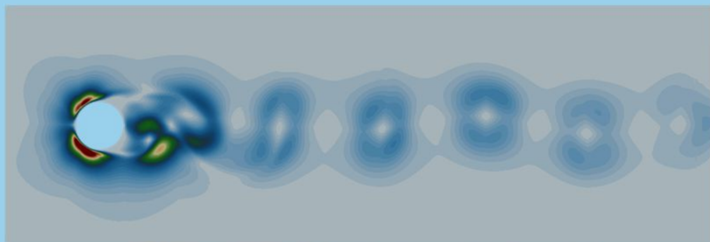


$$\underbrace{\mathbb{BDM}_k - \mathbb{P}^{\mathrm{dc}}_{k-1}}_{\text{pressure-robust}} \qquad \text{vs.} \qquad \mathbb{P}^{\mathrm{dc}}_k - \mathbb{P}^{\mathrm{dc}}_{k-1}$$

coarse grids: non-pressure-robust solvers lose half of (formal) convergence order

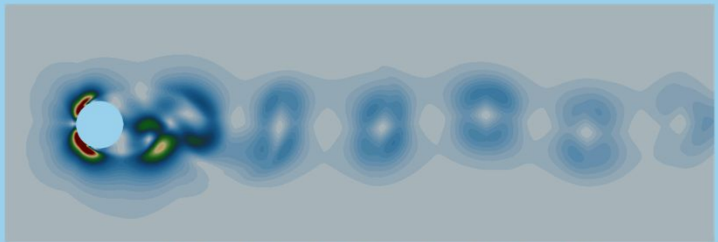
Reference: : N. Gauger, P. Schroeder, A. Linke: arXiv 1808.10711.

Karman vortex street Re=100



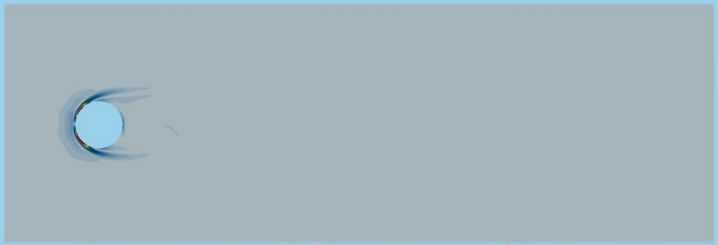
material derivative $|f_h^t = \partial_t u_h + (u_h \cdot \nabla_h) u_h|_p^p$, $p=3/2$

0 5 10 15 20 25 30 35 40 45 50



gradient part of material derivative $|\nabla \phi_h^t|_p^p$, $p=3/2$

0 5 10 15 20 25 30 35 40 45 50



divergence-free part of material derivative $|\mathbb{P}_h^{\mathrm{div}}(f_h^t)|_p^p$, $p=3/2$

0 5 10 15 20 25 30 35 40 45 50

$$\frac{D\mathbf{u}}{\mathrm{Dt}} = -\nabla p + \underbrace{\nu \Delta \mathbf{u}}_{\text{divergence-free}}$$

Reference: N. Gauger, P. Schroeder, A. Linke: arXiv 1808.10711.

- material derivative: small divergence-free part
- pressure-robust schemes: better around obstacle

Classification of pressure-robust solvers

pressure-robustness: H(div)-conforming discretization for incompressible Euler part

- $\mathbf{H}$(div)-conforming DG: G. Kanschat, B. Cockburn, D. Schötzau, J. Schöberl, C. Lehrenfeld, (NGSOLVE !!!), C. Cotter, …
- $\mathbf{H}^1$-conforming ´divergence-free´ schemes: Scott-Vogelius, M. Neilan, J. Guzman, A. Buffa, …
- $\mathbf{H}^1$-conforming ´divergence-free´ IGA: T. Hughes, J. Evans, …
- conforming & non-conforming schemes with $\mathbf{H}$(div)-conforming velocity reconstructions: A. Linke, C. Merdon, L. Tobiska, G. Matthies, A. Ern, D. di Pietro, F. Schieweck, P. Lederer, J. Schöberl, C. Lehrenfeld, W. Wollner, P. Zanotti, C. Kreuzer, R. Verfürth, …

Alternative:

- direct discretization of vorticity equation (in 2d, periodic boundary conditions,)

Messages

Divergence constraint in incompressible flows:

- dominant gradients: source for numerical errors in CFD
- Stokes-inf-sup & BDM-RT-spaces enable pressure-robustness
- CFD: restart out of confusion: possible & necessary

BMBF-VIP+ Proposal

Proposal in funding line VIP+ (‚Validation of technological potential of innovative science') from Federal Ministry of Education and Research (BMBF)

Tentative partners for implementation in SU2:

- Nicolas Gauger, TU Kaiserslautern
- Alexander Linke, Weierstrass Institute, Berlin
- Cornelia Grabe, German Aerospace Center (DLR)

Collaborators (to be confirmed !):

- Edwin van der Weide, U Twente
- Martin Schifko, Engineering Software Steyr (ESS)